



专题 2 裂项相消



秒杀秘籍：第一讲 裂项相消常见推论

第一讲 常见的等差数列与裂项相消：

$$\textcircled{1} a_n = \frac{1}{(An+B)(An+A+B)} = \frac{1}{A} \left(\frac{1}{An+B} - \frac{1}{A(n+1)+B} \right) \quad (\text{接龙型})$$

$$S_n = \frac{1}{A} \left(\frac{1}{A+B} - \frac{1}{A(n+1)+B} \right) = \frac{n}{(A+B)[A(n+1)+B]}$$

$$\textcircled{2} a_n = \frac{1}{(An+B)(An+2A+B)} = \frac{1}{2A} \left(\frac{1}{An+B} - \frac{1}{A(n+2)+B} \right) \quad (\text{隔项型})$$

$$S_n = \frac{1}{2A} \left(\frac{1}{A+B} + \frac{1}{2A+B} - \frac{1}{A(n+1)+B} - \frac{1}{A(n+2)+B} \right)$$

$$\textcircled{3} a_n = \frac{1}{\sqrt{An+B} + \sqrt{A(n+1)+B}} = \frac{1}{A} (\sqrt{A(n+1)+B} - \sqrt{An+B}) \quad (\text{根式型})$$

$$S_n = \frac{1}{A} (\sqrt{A(n+1)+B} - \sqrt{A+B})$$

第二讲 特殊等差数列与裂项相消：

$$a_n = \frac{(-1)^{n-1}(2An+A+2B)}{(An+B)[A(n+1)+B]} = (-1)^{n-1} \left(\frac{1}{An+B} + \frac{1}{A(n+1)+B} \right)$$

$$S_n = \left(\frac{1}{A+B} + \frac{1}{2A+B} \right) - \left(\frac{1}{2A+B} + \frac{1}{3A+B} \right) + \cdots + (-1)^{n-1} \left(\frac{1}{An+B} + \frac{1}{A(n+1)+B} \right) = \frac{1}{A+B} + \frac{(-1)^{n-1}}{A(n+1)+B}$$

第三讲 带有等比数列的裂项相消：

$$a_n = \frac{q^n}{(q^n+m)(q^{n+1}+m)} = \frac{1}{q-1} \left(\frac{1}{q^n+m} - \frac{1}{q^{n+1}+m} \right) \quad (\text{其中 } m \in \mathbb{R}, q \neq 1)$$

$$S_n = \frac{1}{q-1} \left(\frac{1}{q+m} - \frac{1}{q^2+m} + \frac{1}{q^2+m} - \frac{1}{q^3+m} + \cdots + \frac{1}{q^n+m} - \frac{1}{q^{n+1}+m} \right) = \frac{1}{q-1} \left(\frac{1}{q+m} - \frac{1}{q^{n+1}+m} \right)$$

第四讲 平方式递推与裂项相消：

$$\textcircled{1} a_{n+1} = a_n(a_n+1) \Rightarrow \frac{1}{a_{n+1}} = \frac{1}{a_n} - \frac{1}{a_n+1} \Rightarrow \frac{1}{a_n+1} = \frac{1}{a_n} - \frac{1}{a_{n+1}} \Rightarrow \sum_{i=1}^n \frac{1}{a_i+1} = \frac{1}{a_1} - \frac{1}{a_{n+1}}$$

$$\textcircled{2} a_{n+1} = \frac{a_n^2}{m} + a_n \Rightarrow \frac{a_{n+1}}{m} = \frac{a_n}{m} \left(\frac{a_n}{m} + 1 \right) \Rightarrow \frac{1}{a_n+m} = \frac{1}{a_n} - \frac{1}{a_{n+1}} \Rightarrow \sum_{i=1}^n \frac{1}{a_i+m} = \frac{1}{a_1} - \frac{1}{a_{n+1}}$$



$$\textcircled{3} a_{n+1} - 1 = a_n(a_n - 1) \Rightarrow \frac{1}{a_{n+1} - 1} = \frac{1}{a_n - 1} - \frac{1}{a_n} \Rightarrow \frac{1}{a_n} = \frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1} \Rightarrow \sum_{i=1}^n \frac{1}{a_i} = \frac{1}{a_1 - 1} - \frac{1}{a_{n+1} - 1}$$

$$\textcircled{4} a_{n+1} = \frac{a_n^2}{2m} + \frac{m}{2} \Rightarrow \frac{1}{a_{n+1} - m} = \frac{2m}{(a_n - m)(a_n + m)} \Rightarrow \frac{1}{a_n + m} = \frac{1}{a_n - m} - \frac{1}{a_{n+1} - m} \Rightarrow \sum_{i=1}^n \frac{1}{a_i + m} = \frac{1}{a_1 - m} - \frac{1}{a_{n+1} - m}$$

注意：平方式递推，通常题目的设置求和部分的分子会给出裂项相消的方向，通常紧扣分母即可，无需记忆，太多的变形形式子。

第五讲 等差三连项型与裂项相消：

$$\textcircled{1} a_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2} \left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right] \quad \textcircled{2} a_n = \frac{An+B}{n(n+1)(n+2)} = \left[\frac{An+\frac{B}{2}}{n(n+1)} - \frac{A(n+1)+\frac{B}{2}}{(n+1)(n+2)} \right]$$

第六讲 阶乘型与裂项相消： $a_n = \frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$

第七讲 等差与等比混合型： $\textcircled{1} a_n = \frac{q^n[1+(1-q)n]}{n(n+1)} = \frac{q^n}{n} - \frac{q^{n+1}}{n+1}$; $\textcircled{2} a_n = \frac{q^n[k+(1-q^k)n]}{n(n+k)} = \frac{q^n}{n} - \frac{q^{n+k}}{n+k}$

注意：通常采用反推法，就是从右边往左边推导，具体情况将会在例题中说明。

【例1】设数列 $\{a_n\}$ 是首项为 $a_1 > 0$ ，公差 $d > 0$ 的等差数列，求： $S_n = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_n a_{n+1}}$ ；

$$T_n = \frac{a_2}{a_1(a_1+a_2)} + \frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)} + \dots + \frac{a_n}{(a_1+a_2+\dots+a_{n-1})(a_1+a_2+\dots+a_n)}$$

【解析】有已知得 $a_{n+1} - a_n = d$ ，则 $a_{n+1} = a_1 + nd$ $\therefore \frac{1}{a_n a_{n+1}} = \frac{1}{d} \left(\frac{1}{a_n} - \frac{1}{a_{n+1}} \right)$

$$\therefore S_n = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_n a_{n+1}} = \frac{1}{d} \left(\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \dots + \frac{1}{a_n} - \frac{1}{a_{n+1}} \right) = \frac{1}{d} \left(\frac{1}{a_1} - \frac{1}{a_{n+1}} \right) = \frac{n}{a_1 a_{n+1}}$$

$$\begin{aligned} T_n &= \frac{a_2}{a_1(a_1+a_2)} + \frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)} + \dots + \frac{a_n}{(a_1+a_2+\dots+a_{n-1})(a_1+a_2+\dots+a_n)} \\ &= \frac{a_1+a_2-a_1}{a_1(a_1+a_2)} + \frac{a_1+a_2+a_3-(a_1+a_2)}{(a_1+a_2)(a_1+a_2+a_3)} + \dots + \frac{a_1+a_2+\dots+a_n-(a_1+a_2+\dots+a_{n-1})}{(a_1+a_2+\dots+a_{n-1})(a_1+a_2+\dots+a_n)} \\ &= \frac{1}{a_1} - \frac{1}{a_1+a_2} + \frac{1}{a_1+a_2} - \frac{1}{a_1+a_2+a_3} + \dots + \frac{1}{a_1+a_2+\dots+a_{n-1}} - \frac{1}{a_1+a_2+\dots+a_{n-1}+a_n} \\ &= \frac{1}{a_1} - \frac{1}{a_1+a_2+\dots+a_{n-1}+a_n} \end{aligned}$$

【例2】(2019·岳阳二模) 已知数列 $\{a_n\}$ ，若 $a_1 + 2a_2 + \dots + na_n = 2n$ ，则数列 $\{a_n a_{n+1}\}$ 前 n 项和为_____。

【解析】数列 $\{a_n\}$ ，若 $a_1 + 2a_2 + \dots + na_n = 2n$ $\textcircled{1}$ 当 $n \geq 2$ 时， $a_1 + 2a_2 + \dots + (n-1)a_{n-1} = 2(n-1)$ $\textcircled{2}$

$\textcircled{1} - \textcircled{2}$ 得： $na_n = 2n - 2(n-1) = 2$ ，整理得： $a_n = \frac{2}{n}$ ，当 $n=1$ 时， $a_1 = 2$ ，符合通项，故： $a_n = \frac{2}{n}$ ，

所以： $a_n a_{n+1} = \frac{2}{n} \cdot \frac{2}{n+1} = 4 \left(\frac{1}{n} - \frac{1}{n+1} \right)$ ，则： $T_n = 4 \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{n} - \frac{1}{n+1} \right) = 4 \left(1 - \frac{1}{n+1} \right) = \frac{4n}{n+1}$ 。



【例 3】 (2019·武汉期中) 已知数列 $\{a_n\}$ 中, $a_2 a_6 = 64$, 且 $\log_2 a_n, \frac{1}{2} \log_2 a_{n+1}, 1 (n \in \mathbb{N}^*)$ 成等差数列.

(1) 求数列 $\{a_n\}$ 的通项公式;

(2) 若数列 $\{b_n\}$ 满足 $b_n = \frac{a_n}{(a_n+1)(a_{n+1}+1)}$, 数列 $\{b_n\}$ 的前 n 项和为 T_n , 求 T_n .

【解析】 (1) $\because \log_2 a_n, \frac{1}{2} \log_2 a_{n+1}, 1$ 成等差数列, $\therefore 2 \times \frac{1}{2} \log_2 a_{n+1} = \log_2 a_n + 1, \therefore a_{n+1} = 2a_n$, 且 $a_n > 0$,
 \therefore 数列 $\{a_n\}$ 是等比数列, 由 $a_2 a_6 = 64$ 得, $a_4 = 8, \therefore a_1 = 1$, 公比 $q = 2, \therefore a_n = 2^{n-1}$;

(2) 由 (1) 知, $b_n = \frac{2^{n-1}}{(2^{n-1}+1)(2^n+1)} = \frac{1}{2^{n-1}+1} - \frac{1}{2^n+1}$,
 $\therefore T_n = (\frac{1}{2^0+1} - \frac{1}{2^1+1}) + (\frac{1}{2^1+1} - \frac{1}{2^2+1}) + (\frac{1}{2^2+1} - \frac{1}{2^3+1}) + \dots + (\frac{1}{2^{n-2}+1} - \frac{1}{2^{n-1}+1}) + (\frac{1}{2^{n-1}+1} - \frac{1}{2^n+1}) = \frac{1}{2} - \frac{1}{2^n+1}$.

【例 4】 (2019·东莞市期末) 已知数列 $\{a_n\}$ 满足: $a_1 = \frac{1}{2}, a_{n+1} = a_n^2 + a_n$, 用 $[x]$ 表示不超过 x 的最大整数, 则

$[\frac{1}{a_1+1} + \frac{1}{a_2+1} + \dots + \frac{1}{a_{2011}+1}]$ 的值等于 ()

- A. 0 B. 1 C. 2 D. 3

【解析】 又因为 $a_{n+1} = a_n^2 + a_n$, 即 $a_{n+1} - a_n = a_n^2 > 0$, 所以数列是增数列, 并且 $\frac{1}{a_n} > 0$, 又因为 $a_{n+1} = a_n^2 + a_n$,
 即 $a_{n+1} = a_n(1+a_n), \frac{1}{a_{n+1}} = \frac{1}{a_n \cdot (1+a_n)} = \frac{1}{a_n} - \frac{1}{1+a_n}$, 所以 $\frac{1}{a_n+1} = \frac{1}{a_n} - \frac{1}{a_{n+1}}$, 即 $\frac{1}{a_n+1} = \frac{1}{a_n} - \frac{1}{a_{n+1}}$,
 $\frac{1}{a_1+1} + \frac{1}{a_2+1} + \dots + \frac{1}{a_{2011}+1} = \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \dots + \frac{1}{a_{2010}} - \frac{1}{a_{2012}} = \frac{1}{a_1} - \frac{1}{a_{2012}} < \frac{1}{a_1} = 2$,
 $a_1 = \frac{1}{2}, a_2 = \frac{3}{4}, a_3 = \frac{16}{21}, \frac{1}{a_1+1} + \frac{1}{a_2+1} = \frac{2}{3} + \frac{4}{7} > 1$. 所以 $\frac{1}{a_1+1} + \frac{1}{a_2+1} + \dots + \frac{1}{a_{2011}+1} \in (1, 2)$.
 所以 $[\frac{1}{a_1+1} + \frac{1}{a_2+1} + \dots + \frac{1}{a_{2011}+1}] = 1$. 故选 B.

【例 5】 (2018·徐州期末) 在数列 $\{a_n\}$ 中, $a_1 = 2, 2a_{n+1} = a_n^2 + 1, n \in \mathbb{N}^*$, 设 $b_n = \frac{2a_n - 1}{a_n + 1}$, 若数列 $\{b_n\}$ 的前 2018

项和 $S_{2018} > t$, 则整数 t 的最大值为_____.

【解析】 在数列 $\{a_n\}$ 中, $a_1 = 2, 2a_{n+1} = a_n^2 + 1, n \in \mathbb{N}^*$, 可得 $2(a_{n+1} - 1) = a_n^2 - 1 = (a_n - 1)(a_n + 1)$,
 $2a_{n+1} = a_n^2 + 1 \geq 2a_n$, 即有数列 $\{a_n\}$ 递增, 可得 $\frac{1}{2(a_{n+1} - 1)} = \frac{1}{2}(\frac{1}{a_n - 1} - \frac{1}{a_n + 1})$, 即有 $\frac{1}{a_n + 1} = \frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1}$,
 $b_n = \frac{2a_n - 1}{a_n + 1} = 2 - \frac{3}{a_n + 1}$, 则 $S_{2018} = b_1 + b_2 + b_3 + \dots + b_{2018} = 2 \times 2018 - 3(\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \dots + \frac{1}{a_{2018} + 1})$
 $= 4036 - 3(\frac{1}{a_1 - 1} - \frac{1}{a_2 - 1} + \frac{1}{a_2 - 1} - \frac{1}{a_3 - 1} + \dots + \frac{1}{a_{2018} - 1} - \frac{1}{a_{2019} - 1}) = 4036 - 3(1 - \frac{1}{a_{2019} - 1}) = 4033 + \frac{3}{a_{2019} - 1}$,
 而数列 $\{a_n\}$ 递增, $a_1 = 2, a_2 = \frac{5}{2}, a_3 = \frac{29}{8}, a_4 = \frac{841}{128} > 4, \dots, a_{2019} > 4$, 由数列 $\{b_n\}$ 的前 2018 项和 $S_{2018} > t$,
 可得整数 t 的最大值为 4033. 故答案为 4033.



【例6】求和： $S_n = \frac{3}{(1 \times 2)^2} + \frac{5}{(2 \times 3)^2} + \cdots + \frac{2n+1}{[n(n+1)]^2}$.

【解析】设 $a_n = \frac{2n+1}{[n(n+1)]^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$,

则 $S_n = 1 - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots + \frac{1}{n^2} - \frac{1}{(n+1)^2} = 1 - \frac{1}{(n+1)^2} = \frac{2n+1}{(n+1)^2}$.

【例7】求和： $S_n = \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!}$.

【解析】 $\because \frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!} \therefore S_n = \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

【例8】求和： $\frac{4}{1 \times 2 \times 3} + \frac{5}{2 \times 3 \times 4} + \cdots + \frac{n+3}{n(n+1)(n+2)}$.

【解析】 $\because \frac{n+3}{n(n+1)(n+2)} = \frac{1}{2} \left[\frac{2n+3}{n(n+1)} - \frac{2n+5}{(n+1)(n+2)} \right]$

$\therefore \frac{4}{1 \times 2 \times 3} + \frac{5}{2 \times 3 \times 4} + \cdots + \frac{n+3}{n(n+1)(n+2)} = \frac{1}{2} \left(3 - \frac{n+3}{(n+1)(n+2)} \right) = \frac{3n^2 + 8n + 5}{2n^2 + 6n + 4}$.

【例9】求和： $\frac{3}{1 \times 3}, \frac{7}{2 \times 4} \times 3, \frac{11}{3 \times 5} \times 3^2, \dots, \frac{4n-1}{n \times (n+2)} \times 3^{n-1}$.

【解析】本题适合反推： $\because \frac{3^{n-1}}{n} - \frac{3^{n+1}}{n+2} = \frac{-8n+2}{n \times (n+2)} \times 3^{n-1}$, $\therefore \frac{4n-1}{n \times (n+2)} \times 3^{n-1} = -\frac{1}{2} \left(\frac{3^{n-1}}{n} - \frac{3^{n+1}}{n+2} \right)$,

令 $S_n = \frac{3}{1 \times 3} + \frac{7}{2 \times 4} \times 3 + \frac{11}{3 \times 5} \times 3^2 + \cdots + \frac{4n-1}{n \times (n+2)} \times 3^{n-1}$, $\therefore S_n = -\frac{1}{2} \left(1 + \frac{3}{2} - \frac{3n}{n+1} - \frac{3n+1}{n+2} \right)$.

【例10】已知函数 $f(x) = \frac{2x+3}{3x}$, 数列 $\{a_n\}$ 满足 $a_1 = 1, a_{n+1} = f\left(\frac{1}{a_n}\right)$.

(1) 求数列 $\{a_n\}$ 的通项;

(2) 令 $T_n = a_1 a_2 - a_2 a_3 + a_3 a_4 - a_4 a_5 + \cdots + a_{2n-1} a_{2n} - a_{2n} a_{2n+1}$, 求 T_n

(3) 令 $b_n = \frac{1}{a_{n-1} a_n} (n \geq 2), b_1 = 3, S_n = b_1 + b_2 + \cdots + b_n$, 若 $S_n < \frac{m-2000}{2}$ 对于任意的 $n \in N^*$ 都成立, 求最小正整数 m 的值.

【解析】(1) $a_{n+1} = f\left(\frac{1}{a_n}\right) = \frac{\frac{2}{a_n} + 3}{\frac{1}{a_n}} = \frac{2}{a_n} + 3 = a_n + \frac{2}{3}, a_n = \frac{2n+1}{3}$

(2) $T_n = a_1 a_2 - a_2 a_3 + a_3 a_4 - a_4 a_5 + \cdots + a_{2n-1} a_{2n} - a_{2n} a_{2n+1} = a_2(a_1 - a_3) + a_4(a_3 - a_5) + \cdots + a_{2n}(a_{2n-1} - a_{2n+1})$
 $= -\frac{4}{3} \times \frac{n(\frac{5}{3} + \frac{4}{3}n + \frac{1}{3})}{2} = -\frac{4}{9}(2n^2 + 3n) \therefore T_n = -\frac{4}{9}(2n^2 + 3n)$

(3) $b_n = \frac{1}{a_{n-1} a_n} = \frac{9}{(2n-1)(2n+1)} (n \geq 2), \because b_1 = 3, \therefore b_n = \frac{9}{(2n-1)(2n+1)}, \therefore S_n = \frac{9n}{2n+1} < \frac{m-2000}{2}$ 对于任意的 $n \in N^*$ 都成立, $\therefore S_n = \frac{9n}{2n+1} \in \left[3, \frac{9}{2}\right), \therefore \frac{m-2000}{2} \geq \frac{9}{2} \therefore m \geq 2009$.



达标训练

1. (2019·思明月考) 设满足 $a_1 + \frac{1}{3}a_2 + \frac{1}{5}a_3 + \dots + \frac{1}{2n-1}a_n = n$.

(1) 求数列 $\{a_n\}$ 的通项公式;

(2) 求数列 $\left\{\frac{1}{\sqrt{a_{n+1}} + \sqrt{a_n}}\right\}$ 的前 84 项和.

2. 已知数列 $\{a_n\}$ 前 n 项和 S_n , 点 $\left(n, \frac{S_n}{n}\right)$ 在直线 $y = \frac{1}{2}x + \frac{11}{2}$ 上; 数列 $\{b_n\}$ 满足 $b_{n+2} - 2b_{n+1} + b_n = 0$, 且 $b_3 = 11$,

前 9 项和为 153.

(1) 求数列 $\{a_n\}$, $\{b_n\}$ 的通项;

(2) 设 $c_n = \frac{3}{(2a_n - 11)(2b_n - 1)}$, 数列 $\{c_n\}$ 前 n 项和 T_n , 求使不等式 $T_n > \frac{k}{57}$ 对于任意的 $n \in N^*$ 都成立的最大正整数 k 的值.

3. (2018·云阳期末) 已知数列 $\{a_n\}$ 满足: $a_1 = \frac{1}{2}$, $a_2 = 1$, $a_{n+1} = a_n + a_{n-1} (n \in N^*, n \geq 2)$, 则

$\frac{1}{a_1 a_3} + \frac{1}{a_2 a_4} + \frac{1}{a_3 a_5} + \dots + \frac{1}{a_{2018} a_{2020}}$ 的整数部分为 ()

- A. 0 B. 1 C. 2 D. 3

4. (2019·韶关模拟) 已知数列 $\{a_n\}$ 满足 $a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \dots + \frac{1}{n}a_n = n^2 + n (n \in N^*)$, 设数列 $\{b_n\}$ 满足: $b_n = \frac{2n+1}{a_n a_{n+1}}$,

数列 $\{b_n\}$ 的前 n 项和为 T_n , 若 $T_n < \frac{n}{n+1} \lambda (n \in N^*)$ 恒成立, 则实数 λ 的取值范围为 ()

- A. $[\frac{1}{4}, +\infty)$ B. $(\frac{1}{4}, \infty)$ C. $[\frac{3}{8}, +\infty)$ D. $(\frac{3}{8}, +\infty)$

5. 已知 $a_n = \frac{n^2}{(2n-1)(2n+1)}$, 求 $\{a_n\}$ 的前 n 项和 S_n .



6. 求和: $\frac{3}{1!+2!+3!} + \frac{4}{2!+3!+4!} + \cdots + \frac{n+2}{n!+(n+1)!+(n+2)!}$.

7. 已知数列 $\{a_n\}$ 通项公式 $a_n = \frac{5n+4}{n(n+1)(n+2)}$, 其前 n 项和 S_n , 是否存在常数 a, b , 使得 $S_n = \frac{an^2+bn}{2(n+1)(n+2)}$ 对于任意的 $n \in N^*$ 都成立? 证明你的结论.

8. 求和: $S_n = \frac{3}{1 \times 2} \cdot \frac{1}{2} + \frac{7}{2 \times 3} \cdot \frac{1}{2^2} + \cdots + \frac{n+2}{n \times (n+1)} \cdot \frac{1}{2^n}$.

9. (2019·长沙月考) 设数列 $\{a_n\}$ 的前 n 项和为 S_n , 若 $a_n - \frac{S_n}{2} = 1 (n \in N^*)$.

(1) 求出数列 $\{a_n\}$ 的通项公式;

(2) 已知 $b_n = \frac{2^n}{(a_n-1)(a_{n+1}-1)} (n \in N^*)$, 数列 $\{b_n\}$ 的前 n 项和记为 T_n , 证明: $T_n \in [\frac{2}{3}, 1)$.

10. (2019·黄山二模) 已知数列 $\{a_n\}$ 满足 $\frac{1}{a_1-1} + \frac{2}{a_2-1} + \frac{3}{a_3-1} + \cdots + \frac{n}{a_n-1} = n, n \in N^*$.

(1) 求数列 $\{a_n\}$ 的通项公式;

(2) 令 $b_n = \frac{2n+1}{(a_n-1)^2(a_{n+1}-1)^2}$, 数列 $\{b_n\}$ 的前 n 项和为 T_n , 求证: $T_n < 1$.



11. (2019•蚌山月考) 已知数列 $\{a_n\}$ 满足 $a_{n+1} + 1 = \frac{a_n + 1}{a_n + 2}$, $a_n \neq -1$ 且 $a_1 = 1$.

(1) 求证: 数列 $\left\{\frac{1}{a_n + 1}\right\}$ 是等差数列, 并求出数列 $\{a_n\}$ 的通项公式;

(2) 令 $b_n = a_n + 1$, $c_n = (-1)^{n-1} n b_n b_{n+1}$, 求数列 $\{c_n\}$ 的前 2019 项和 S_{2019} .

12. (2019•郑州二模) 数列 $\{a_n\}$ 满足: $\frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_n}{n+1} = n^2 + n$, $n \in \mathbb{N}^*$.

(1) 求 $\{a_n\}$ 的通项公式;

(2) 设 $b_n = \frac{1}{a_n}$, 数列 $\{b_n\}$ 的前 n 项和为 S_n , 求满足 $S_n > \frac{9}{20}$ 的最小正整数 n .

13. (2019•涪城模拟) 已知等比数列 $\{a_n\}$ 的前 n 项和是 S_n , 且 $S_n = 2^{n+1} - b$.

(1) 求 b 的值及数列 $\{a_n\}$ 的通项公式;

(2) 令 $b_n = \frac{a_n}{(a_n - 1)(a_{n+1} - 1)}$, 数列 $\{b_n\}$ 的前 n 项和 T_n , 证明: $T_n \geq \frac{2}{3}$.



14. 已知数列 $\{a_n\}$ 中, $a_1 = 2$, 若 $a_{n+1} - a_n = a_n^2$, 设 $T_m = \frac{a_1}{a_1+1} + \frac{a_2}{a_2+1} + \dots + \frac{a_m}{a_m+1}$, 若 $T_m < 2018$, 则正整数 m 的最大值为 ()
- A. 2019 B. 2018 C. 2017 D. 2016
15. (2019·河南月考) 数列 $\{a_n\}$ 满足 $a_1 = \frac{6}{5}$, $a_n = \frac{a_{n+1}-1}{a_n-1} (n \in N^*)$, 若对 $n \in N^*$, 都有 $k > \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$ 成立, 则最小的整数 k 是 ()
- A. 3 B. 4 C. 5 D. 6
16. (2018·渝水月考) 已知数列 $\{a_n\}$ 满足 $a_1 = \frac{4}{3}$, 且 $a_{n+1} - 1 = a_n(a_n - 1) (n \in N^*)$, 则 $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{2017}}$ 的整数部分是 ()
- A. 0 B. 1 C. 2 D. 3
17. (2018·历下月考) 用 $[x]$ 表示不超过 x 的最大整数, 例如 $[3] = 3$, $[1.2] = 1$, $[-1.3] = -2$. 已知数列 $\{a_n\}$ 满足 $a_1 = 1$, $a_{n+1} = a_n^2 + a_n$, 则 $[\frac{a_1}{1+a_1} + \frac{a_2}{1+a_2} + \dots + \frac{a_{2018}}{1+a_{2018}}] =$ _____.
18. (2019·武汉模拟) 数列 $\{a_n\}$ 满足 $a_1 = \frac{3}{2}$, $a_{n+1} = a_n^2 - a_n + 1 (n \in N^*)$, 则 $m = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{2019}}$ 的整数部分是 ()
- A. 0 B. 1 C. 2 D. 3
19. (2018·虎林模拟) 数列 $\{a_n\}$ 满足 $a_1 = \frac{4}{3}$, $a_{n+1} - 1 = a_n(a_n - 1) (n \in N^*)$, 且 $S_n = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$, 则 S_n 的整数部分的所有可能值构成的集合是 ()
- A. $\{0, 1, 2\}$ B. $\{0, 1, 2, 3\}$ C. $\{1, 2\}$ D. $\{0, 2\}$
20. (2019·湖州模拟) 已知数列 $\{a_n\}$ 满足 $a_1 = \frac{1}{2}$, $a_{n+1} = \frac{a_n^2}{2018} + a_n (n \in N^*)$, 则使 $a_n > 1$ 的正整数 n 的最小值是 ()
- A. 2018 B. 2019 C. 2020 D. 2021
21. (2019·浙江期中) 已知数列 $\{a_n\}$ 满足: $a_1 = 3$, $2a_{n+1} = a_n^2 - 2a_n + 4$.

(1) 求证: $a_{n+1} > a_n$;

(2) 求证: $\frac{1}{3} \leq \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n} \leq 1 - (\frac{2}{3})^n (n \in N^*)$.



22. 已知数列 $\{a_n\}, \{b_n\}$ 满足, $a_1 = \frac{1}{2}, b_1 = \frac{1}{2}$, 且对于任意的 $m, n \in \mathbb{N}^*$, 有 $a_{m+n} = a_m \cdot a_n, b_{m+n} = b_m + b_n$.

(1) 求数列 $\{a_n\}, \{b_n\}$ 的通项;

(2) 设 $b_n = \frac{4c_n + n}{3c_n + n}$, 求 $\{c_n\}$ 的通项;

(3) 若数列 $\{d_n\}$ 满足 $d_n = \frac{a_n}{c_n}$, 其前 n 项和 T_n , 求证: $n \geq 2$ 时, $-\frac{5}{2} < T_n < a_n - \frac{5}{2}$.

23. 数列 $\{a_n\}$ 满足 $a_{n+1} = \frac{(n+1)(2a_n - n)}{a_n + 4n}$, 且 $a_1 = \frac{1}{2}$.

(1) 求 a_2, a_3, a_4 ;

(2) 若存在实数 a , 使数列 $\left\{ \frac{a_n + an}{a_n + n} \right\}$ 成为以 -1 为公差的等差数列, 求实数 a ;

(3) 记数列 $\left\{ \frac{1}{3^{\frac{n+2}{2}} a_{n+2}} \right\}$ 的前 n 项和 S_n , 求证: $S_n > -\frac{2\sqrt{3}+1}{12}$.



24. (2019·静安一模) 将 n 个数 a_1, a_2, \dots, a_n 的连乘积 $a_1 \cdot a_2 \cdot \dots \cdot a_n$ 记为 $\prod_{i=1}^n a_i$, 将 n 个数 a_1, a_2, \dots, a_n 的和 $a_1 + a_2 + \dots + a_n$ 记为 $\sum_{i=1}^n a_i$, $n \in N^*$).

(1) 若数列 $\{x_n\}$ 满足 $x_1 = 1$, $x_{n+1} = x_n^2 + x_n$, $n \in N^*$, 设 $P_n = \prod_{i=1}^n \frac{1}{1+x_i}$, $S_n = \sum_{i=1}^n \frac{1}{1+x_i}$, 求 $P_5 + S_5$;

(2) 用 $[x]$ 表示不超过 x 的最大整数, 例如 $[2] = 2$, $[3.4] = 3$, $[-1.8] = -2$. 若数列 $\{x_n\}$ 满足 $x_1 = 1$, $x_{n+1} = x_n^2 + x_n$, $n \in N^*$, 求 $[\sum_{i=1}^{2019} \frac{x_i}{1+x_i}]$ 的值;

(3) 设定义在正整数集 N^* 上的函数 $f(n)$ 满足, 当 $\frac{m(m-1)}{2} < n \leq \frac{m(m+1)}{2}$ ($m \in N^*$) 时, $f(n) = m$, 问是否存在正整数 n , 使得 $\sum_{i=1}^n f(i) = 2019$? 若存在, 求出 n 的值; 若不存在, 说明理由 (已知 $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$).